# Properties for Allocation Rules for Cooperative Games and Revenue Sharing Problems

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# Chapter 7

# Efficient TU-game solutions based on revenue sharing rules

## 7.1 Introduction

When O'Neill (1982, [66]) constructed the model of a bankruptcy problem, he also associated with each bankruptcy problem a cooperative game to describe the relationship between the rule for bankruptcy problems and the solution for cooperative games. He set the worth of each coalition $S$ equal to the difference between the amount available and the sum of the claims of the members of the complementary coalition, $N \backslash S$, if this difference is non-negative, and 0 otherwise. The difference can be understood as what the complementary coalition "concedes" to $S$. It is what $S$ can secure for itself. Formally, to a bankruptcy problem $(N, E, c) \in \mathcal{B}^N$, we assign a *coalitional game* $\langle N, v_{E,c} \rangle$ whose characteristic function is given by

$$v_{E,c}(S) = \max\{E - \sum_{j \in N \backslash S} c_j, 0\}, \text{ for all } S \subseteq N.$$

Based on this coalitional game, the next theorems link some bankruptcy rules to solutions of coalitional games. In the following, all rules and properties we don't define in this chapter can see Chapter 1, Section 1.2 for the definitions.

**Theorem 7.1** (O'Neill 1982, [66]). *The random arrival bankruptcy rule of a bankruptcy problem is the Shapley value of the associated coalitional game.*

**Theorem 7.2** (Aumann and Maschler 1985, [6]). *The Talmud bankruptcy rule of a bankruptcy problem is the prenucleolus of the associated coalitional game.*

**Theorem 7.3** (Curiel et al. 1987, [15]). *The adjusted proportional bankruptcy rule of a bankruptcy problem is the $\tau$ value of the associated coalitional game.*

Herrero et al. (1999, [38]) only considered the efficiency requirement and ignored the individual rationality to define division for bankruptcy problems. The *rights-egalitarian bankruptcy solution* is defined by, for each $(N, E, c) \in \mathcal{B}^N$ and $i \in N$,

$$RE_i(N, E, c) = c_i + \frac{1}{n}\big(E - \sum_{j \in N} c_j\big).$$

To further dicuss the rights-egalitarian bankruptcy solution, Herrero et al. (1999, [38]) built two new coalitional games for bankruptcy problems. Given a bankruptcy problem $(N, E, c) \in \mathcal{B}^N$, its *direct coalitional game* is the game $\langle N, v_D \rangle$ whose characteristic function is given by

$$v_D(S) = \begin{cases} \sum_{i \in S} c_i, & \text{if } S \neq N; \\ E, & \text{if } S = N. \end{cases}$$

Its *complementary coalitional game* is the game $\langle N, v_C \rangle$ whose characteristic function is given by

$$v_C(S) = \begin{cases} E - \sum_{i \in N \setminus S} c_i, & \text{if } S \neq \emptyset; \\ 0, & \text{if } S = \emptyset. \end{cases}$$

Then, Herrero et al. gave out the relationship between the rights-egalitarian allocation rule and the Shapley value.

**Theorem 7.4** (Herrero et al. 1999, [38]). *The rights-egalitarian allocation rule of a bankruptcy problem is the Shapley value for both the direct coalitional game and the complementary coalitional game.*

These studies established the bridge between the rules of bankruptcy problems and the values in cooperative games. Apparently, they only considered these efficient values in cooperative games, because the insufficient estate should be divided entirely among all claimants.

## 7.2 Values without efficiency

The study of methods for measuring the "value" of playing a particular role in cooperative games is motivated by several considerations. One is to determine an equitable distribution of the wealth available to the players through their participation in the game. Another is to help an individual assess his prospects from participation in the game.

When a method of valuation is used to determine equitable distributions, a natural defining property is "efficiency". However, when the players of a game individually assess their powers in the game, there is no reason to suppose that these assessments will be jointly efficient.

The Deegan-Packel value (Deegan and Packel 1978, [17]), proposed as a power index by considering the average worth of coalition with the player, is not efficient. For any game $\langle N, v\rangle \in \mathcal{G}^N$, the Deegan-Packel value is given by

$$DP_i(N, v) = \sum_{S \subseteq N, S \ni i} \frac{v(S)}{s}, \text{ for all } i \in N.$$

In Deegan and Packel's characterization, the value satisfies

$$\sum_{i \in N} DP_i(N, v) = \sum_{S \subseteq N} v(S). \tag{7.1}$$

The Banzhaf value (Banzhaf 1965, [8]), introduced as a power index for voting games, is also not efficient. The *semivalues* (Dubey *et al.* 1981, [24]), defined as the generalizations and analogues of the Shapley value, is of the form

$$SE_i(N,v) = \sum_{S\subseteq N, S\ni i} p_s^n[v(S) - v(S\backslash\{i\})], \text{ for all } i \in N,$$

where $p^n = (p_s^n)_{s=1}^n$ is a collection of nonnegative real numbers satisfying the normalization condition

$$\sum_{s=1}^{n} \binom{n-1}{s-1} p_s^n = 1.$$

That is, $p^n$ is a probability distribution over coalitions containing player $i$, which assigns the same probability to coalitions of the same size. Thus, a semivalue allocates to each player the expected marginal contribution based on the probability distribution $p^n$. Clearly, for

$$p_s^n = \frac{(s-1)!(n-s)!}{n!}, \quad s = 1, 2, \ldots, n,$$

we get precisely the Shapley value. In fact, the Shapley value is the unique efficient semivalue. The Banzhaf value is the semivalue with a uniform probability distribution

$$p_s^n = \frac{1}{2^{n-1}}, \quad s = 1, 2, \ldots, n.$$

**Theorem 7.5** (Dubey *et al.* 1981, [24])**.** *Any value $\phi$ on $\mathcal{G}^N$ possesses linearity, symmetry, monotonicity and the inessential game property if and only if $\phi$ is a semivalue.*

Based on the inefficient power index, how to set up the corresponding efficient allocations? In the existing literatures, there are mainly two methods. One is multiplicative efficient normalization. The *multiplicative efficient normalized Banzhaf value* (Dubey and Shapley 1979, [25]) is given as[1]

$$\varphi_i^B(N,v) = \frac{Ban_i(N,v)}{\sum_{j\in N} Ban_j(N,v)} v(N), \text{ for all } i \in N.$$

[1]Notice that this is only defined if the sum of the Banzhaf value payoffs of the players is nonzero.

The multiplicative efficient normalized Banzhaf value distributes the worth of the grand coalition with the individual probability numbered as the corresponding Banzhaf value in the sum of all players' Banzhaf values. This still satisfies symmetry and monotonicity, but doesn't satisfy linearity and the inessential game property.

The other one is additive efficient normalization. For any semivalue $SE(N, v)$, the *additive efficient normalized semivalue* (Ruiz et al. 1998, [77]) is defined as

$$\varphi_i^{SE}(N, v) = SE_i(N, v) + \frac{1}{n}\big[v(N) - \sum_{j \in N} SE_j(N, v)\big], \text{ for all } i \in N.$$

The additive efficient normalized semivalue assigns to each player his or her expected marginal contribution $SE_i(N, v)$, $i \in N$, and then divides the remainder (positive or negative), equally among all the players. This still satisfies linearity, symmetry and monotonicity, but doesn't satisfy the inessential game property.

With the two methods, it is easy to build the multiplicative efficient normalized Deegan-Packel value, the multiplicative efficient normalized semivalues and the additive efficient normalized Deegan-Packel value. In this Chapter, we try to normalize these inefficient values with the revenue sharing rules in the revenue problems.

## 7.3 Normalizations of values

According to the equation (7.1), for any game $\langle N, v\rangle \in \mathcal{G}^N$, the sum of all players' Deegan-Packel values, $\sum_{i \in N} DP_i(N, v)$, is usually greater than the worth of the grand coalition, $v(N)$. Although the semivalues are not all efficient, it is not clear to compare the size between the worth of the grand coalition and the sum of all players' payoff in these inefficient values. In Chapter 6, the revenue problems describe investment situations where the relationship between the revenue and the sum of all investments is not explicit. It is natural to establish the connection for the inefficient values and the revenue sharing rules in the revenue problems.

A revenue problem is a triple $(N, R, d)$, where $N$ is the set of investors, $R \in \mathbb{R}_+$ represents the total revenue of the enterprise, and $d = \{d_1, d_2, \ldots, d_n\} \in \mathbb{R}^N_+$ is a vector of investments. For any game $\langle N, v\rangle \in \mathcal{G}^N$ and inefficient value $\phi(N, v)$, the set of investors should be $N$, and the total revenue is $R = v(N)$. However, it's difficult to guarantee that $\phi_i(N, v)$ for all $i \in N$ is non-negative. Moreover, for multiplicative efficient normalization is not guaranteed that the sum of payoffs over all players is nonzero. Hence, in the following, we only focus on the positive games and inefficient positive values.

A game $\langle N, v\rangle \in \mathcal{G}^N$ is *positive*, if

$$v(S) > 0, \text{ for all } S \in \Omega_N.$$

Denote by $\mathcal{P}^N$ the set of all positive games. A *positive value* $\phi$ is a function that assigns a positive payoff vector $(\phi_i(N, v))_{i\in N} \in \mathbb{R}^N_{++}$ to every positive game $\langle N, v\rangle \in \mathcal{P}^N$ (see Ortmann 2000, [68]).

For any positive game $\langle N, v\rangle \in \mathcal{P}^N$ and inefficient positive value $\phi : \mathcal{P}^N \to \mathbb{R}^N_{++}$, the *corresponding revenue problem* $RP^\phi_v$ is defined as a triple $(N, v(N), \phi)$, where $\phi = \{\phi_i(N, v)\}_{i\in N}$.

**Definition 7.6.** For any positive game $\langle N, v\rangle \in \mathcal{P}^N$ and any revenue sharing rule $f$ for revenue problems, the *$f$-normalized value* $FV(N, v)$ of any inefficient positive value $\phi(N, v)$ is given by

$$FV^\phi_i(N, v) = f_i(N, v(N), \phi), \text{ for all } i \in N.$$

In all revenue sharing rules, the proportional revenue sharing rule is the most commonly used allocation method. For any positive game $\langle N, v\rangle \in \mathcal{P}^N$ and inefficient positive value $\phi : \mathcal{P}^N \to \mathbb{R}^N_{++}$, the *proportional-normalized value*, $PV^\phi(N, v)$, is defined as

$$PV^\phi_i(N, v) = \frac{\phi_i(N, v)}{\sum_{j\in N} \phi_j(N, v)} v(N) = P_i(N, v(N), \phi), \text{ for all } i \in N.$$

The definition of revenue sharing rule for revenue problems, requires that a revenue sharing rule must satisfy three conditions: efficiency, individual rationality and investor rationality. In reality, when we pursue more equitable, it is hard to take account of the rationality. That is to say, when social resources are constant, we cannot balance the resources of all people without reducing the benefits of others. Therefore, we omit the rationality conditions, then we can derive the equal surplus allocation for the revenue problem, which is similar as the rights-egalitarian bankruptcy solution of the bankruptcy problem.

For each $(N, R, d) \in \mathcal{R}^N$, the *equal surplus allocation* is defined as

$$ESA_i(N, R, d) = d_i + \frac{1}{n}(R - \sum_{j \in N} d_j), \text{ for all } i \in N.$$

For any positive game $\langle N, v \rangle \in \mathcal{P}^N$ and inefficient positive value $\phi : \mathcal{P}^N \to \mathbb{R}^N_{++}$, the *equal-surplus-normalized value, $ESV^\phi(N, v)$*, is defined as, for all $i \in N$,

$$ESV_i^\phi(N, v) = \phi_i(N, v) + \frac{1}{n}\big[v(N) - \sum_{j \in N} \phi_j(N, v)\big] = ESA_i(N, v(N), \phi).$$

## 7.4 Properties

Efficiency is the target to normalize the inefficient positive values, so it is apparent that all the normalizations of the values are efficient. In the following, we will first discuss those properties described in Chapter 1, then study two balanced contributions properties.

As a first analysis, for any inefficient positive value $\phi : \mathcal{P}^N \to \mathbb{R}^N_{++}$, assuming that $\phi$ satisfies those properties listed in Section 1.1.3 for positive games, the following table will show whether or not its proportional-normalized value, $PV^\phi$, and equal-surplus-normalized value, $ESV^\phi$, satisfy those properties for positive games, where $+$ denotes satisfy, and $-$ denotes not satisfy.

**Properties**

| Property for $\mathcal{P}^N$ | $PV^\phi$ | $ESV^\phi$ |
|---|---|---|
| Symmetry | $+$ | $+$ |
| Additivity | $-$ | $+$ |
| Translation covariance | $-$ | $+$ |
| Linearity | $-$ | $+$ |
| Null player in a productive environment | $+$ | $-$ |
| Fairness | $-$ | $+$ |
| Inessential game property | $+$ | $+$ |
| Continuity | $+$ | $+$ |
| Monotonicity | $+$ | $-$ |
| Coalitional monotonicity | $+$ | $-$ |
| Desirability | $+$ | $-$ |
| Strong monotonicity | $-$ | $-$ |
| Weak monotonicity | $+$ | $-$ |

Note that, the proportional-normalized value and the equal-surplus-normalized value both satisfy symmetry, continuity, and the inessential game property. It is easy to understand that if an inefficient positive value is symmetric or continuous, then its two normalized values are also symmetric or continuous. If an inefficient positive value $\phi$ satisfies the inessential game property, then for any positive inessential game $\langle N, v\rangle$, $\phi(N, v)$ must be efficient. Therefore, the two normalized values satisfy the inessential game property.

Because the proportional-normalized value is not linear, it doesn't satisfy additivity, translation covariance and linearity, while the equal-surplus-normalized value satisfies these properties. This is also the reason that the proportional-normalized value doesn't satisfy fairness, but the equal-surplus-normalized value satisfies it, even though the two normalized values are symmetric.

When a property relates to comparing one player's payoff with a value, another player's payoff or his payoff in another game, such as the null player in a productive environment, desirability, coalitional monotonicity, strong monotonicity and weak monotonicity, the equal-surplus-normalized value doesn't satisfy these properties. As although an inefficient positive value, $\phi$, satisfies these properties, for any positive game $\langle N, v\rangle \in \mathcal{P}^N$, we can't ensure that the difference, $v(N) - \sum_{j\in N}\phi_j(N,v)$, must be non-negative. However, for any positive game $\langle N, v\rangle \in \mathcal{P}^N$ and inefficient positive value $\phi : \mathcal{P}^N \to \mathbb{R}^N_{++}$, the sum of all players' payoff, $\sum_{j\in N}\phi_j(N,v)$, must be positive. Hence, the proportional-normalized value satisfies these properties except strong monotonicity in positive games. The proportional-normalized value can't satisfy strong monotonicity because we can't compare the worths of the grand coalition in any pair of positive games.

Myerson (1980, [60]) considerd each player's contribution to the other players in games by the difference in the payoffs of a given game and the games that the player leaves. Let $\phi : \mathcal{G}^N \to \mathbb{R}^N$ be a value. Player $j$'s *contribution* to $i$ in $\langle N, v\rangle$ is

$$\phi_i(N,v) - \phi_i(N\backslash\{j\}, v_{-j}),$$

where $\langle N, v_{-j}\rangle$ is defined as $v_{-j}(S) = v(S)$ for all $S \subseteq N\backslash\{j\}$. A value $\phi$ satisfies balanced contributions, if $\phi$ balances contributions between any two players.

- *Balanced contributions*: For any game $\langle N, v\rangle \in \mathcal{G}^N$, the value satisfies $\phi_i(N,v) - \phi_i(N\backslash\{j\}, v_{-j}) = \phi_j(N,v) - \phi_j(N\backslash\{i\}, v_{-i})$ for any pair of players $i$, $j \in N$.

**Proposition 7.7.** *Given any inefficient positive value $\phi : \mathcal{P}^N \to \mathbb{R}^N_{++}$ satisfying balanced contributions in $\mathcal{P}^N$, its equal-surplus-normalized value $ESV^\phi$ satisfies balanced contributions in $\mathcal{P}^N$, if and only if, $\phi$ satisfies, for any positive game $\langle N, v\rangle \in \mathcal{P}^N$ and any pair of players $i$, $j \in N$,*

$$\sum_{k\in N\backslash\{j\}} \phi_k(N\backslash\{j\}, v_{-j}) - v_{-j}(N\backslash\{j\}) = \sum_{k\in N\backslash\{i\}} \phi_k(N\backslash\{i\}, v_{-i}) - v_{-i}(N\backslash\{i\}).$$

**Proof.** For any positive game $\langle N, v\rangle \in \mathcal{P}^N$, let $\phi : \mathcal{P}^N \to \mathbb{R}^N_{++}$ be an inefficient positive value satisfying balanced contributions in $\mathcal{P}^N$, *i.e.*,

$$\phi_i(N,v) - \phi_i(N\backslash\{j\}, v_{-j}) = \phi_j(N,v) - \phi_j(N\backslash\{i\}, v_{-i}). \tag{7.2}$$

The corresponding equal-surplus-normalized value $ESV^{\phi}$ satisfies balanced contributions in $\mathcal{P}^N$, if and only if, for any positive game $\langle N, v\rangle \in \mathcal{P}^N$ and any pair of players $i$, $j \in N$,

$$ESV_i^{\phi}(N,v) - ESV_i^{\phi}(N\backslash\{j\}, v_{-j}) = ESV_j^{\phi}(N,v) - ESV_j^{\phi}(N\backslash\{i\}, v_{-i}). \tag{7.3}$$

According to the definition of the equal-surplus-normalized value, Equation (7.3) is equivalent to

$$\begin{aligned}
&\phi_i(N,v) + \frac{1}{n}\big[v(N) - \sum_{k\in N}\phi_k(N,v)\big] \\
&\quad - \phi_i(N\backslash\{j\}, v_{-j}) - \frac{1}{n-1}\big[v_{-j}(N\backslash\{j\}) - \sum_{k\in N\backslash\{j\}}\phi_k(N\backslash\{j\}, v_{-j})\big] \\
&= \phi_j(N,v) + \frac{1}{n}\big[v(N) - \sum_{k\in N}\phi_k(N,v)\big] \\
&\quad - \phi_j(N\backslash\{i\}, v_{-i}) - \frac{1}{n-1}\big[v_{-i}(N\backslash\{i\}) - \sum_{k\in N\backslash\{i\}}\phi_k(N\backslash\{i\}, v_{-i})\big]
\end{aligned}$$

$$\overset{\text{Eq (7.2)}}{\Leftrightarrow}$$

$$\begin{aligned}
&\sum_{k\in N\backslash\{j\}}\phi_k(N\backslash\{j\}, v_{-j}) - v_{-j}(N\backslash\{j\}) \\
&= \sum_{k\in N\backslash\{i\}}\phi_k(N\backslash\{i\}, v_{-i}) - v_{-i}(N\backslash\{i\}).
\end{aligned}$$

This completes the proof. $\square$

Let $\phi : \mathcal{G}^N \to \mathbb{R}^N$ be an inefficient value. For any game $\langle N, v\rangle \in \mathcal{G}^N$, we define the *distribution difference* of $\phi$ in $\langle N, v\rangle$ as

$$\sum_{k\in N}\phi_k(N,v) - v(N).$$

Proposition 7.7 can be identified as saying that the equal-surplus-normalized value of an inefficient positive value satisfying balanced contributions in $\mathcal{P}^N$ satisfies balanced contributions in $\mathcal{P}^N$, if and only if, for any positive game $\langle N, v\rangle \in \mathcal{P}^N$ and any pair of players $i$, $j \in N$, the inefficient positive value's distribution differences in $\langle N, v_{-j}\rangle$ and $\langle N, v_{-i}\rangle$ are equal.

Ortmann (2000, [68]) introduced an alternative approach to define a player's contribution to the others. Let $\phi : \mathcal{P}^N \to \mathbb{R}^N_{++}$ be a positive value. He defined player $j$ 's *relative contribution* to player $i$ by the quotient

$$\frac{\phi_i(N, v)}{\phi_i(N\backslash\{j\}, v_{-j})}.$$

A value $\phi$ satisfies balanced relative contributions, if $\phi$ balances relative contributions between any two players.

- *Balanced relative contributions*: For any positive game $\langle N, v\rangle \in \mathcal{P}^N$, the positive value satisfies $\dfrac{\phi_i(N, v)}{\phi_i(N\backslash\{j\}, v_{-j})} = \dfrac{\phi_j(N, v)}{\phi_j(N\backslash\{i\}, v_{-i})}$ for any pair of players $i$, $j \in N$.

**Proposition 7.8.** *Given any inefficient positive value $\phi : \mathcal{P}^N \to \mathbb{R}^N_{++}$ satisfying balanced relative contributions in $\mathcal{P}^N$, its proportional-normalized value $PV^\phi$ satisfies balanced relative contributions in $\mathcal{P}^N$, if and only if, $\phi$ satisfies, for any positive game $\langle N, v\rangle \in \mathcal{P}^N$ and any pair of players $i$, $j \in N$,*

$$\frac{\sum_{k\in N\backslash\{j\}} \phi_k(N\backslash\{j\}, v_{-j})}{v_{-j}(N\backslash\{j\})} = \frac{\sum_{k\in N\backslash\{i\}} \phi_k(N\backslash\{i\}, v_{-i})}{v_{-i}(N\backslash\{i\})}.$$

**Proof.** For any positive game $\langle N, v\rangle \in \mathcal{P}^N$, let $\phi : \mathcal{P}^N \to \mathbb{R}^N_{++}$ be an inefficient positive value satisfying balanced relative contributions in $\mathcal{P}^N$, *i.e.*,

$$\frac{\phi_i(N, v)}{\phi_i(N\backslash\{j\}, v_{-j})} = \frac{\phi_j(N, v)}{\phi_j(N\backslash\{i\}, v_{-i})}. \tag{7.4}$$

The corresponding proportional-normalized value $PV^\phi$ satisfies balanced relative contributions in $\mathcal{P}^N$, if and only if, for any positive game $\langle N, v\rangle \in \mathcal{P}^N$ and any pair of players $i$, $j \in N$,

$$\frac{PV_i^\phi(N, v)}{PV_i^\phi(N\backslash\{j\}, v_{-j})} = \frac{PV_j^\phi(N, v)}{PV_j^\phi(N\backslash\{i\}, v_{-i})}. \tag{7.5}$$

According to the definition of the proportional-normalized value, Equation (7.5) is equivalent to

$$\frac{\phi_i(N,v)}{\sum_{k\in N}\phi_k(N,v)}v(N)\bigg/\frac{\phi_i(N\backslash\{j\},v_{-j})}{\sum_{k\in N\backslash\{j\}}\phi_k(N\backslash\{j\},v_{-j})}v_{-j}(N\backslash\{j\})$$
$$=\frac{\phi_j(N,v)}{\sum_{k\in N}\phi_k(N,v)}v(N)\bigg/\frac{\phi_j(N\backslash\{i\},v_{-i})}{\sum_{k\in N\backslash\{i\}}\phi_k(N\backslash\{i\},v_{-i})}v_{-i}(N\backslash\{i\})$$
$$\Leftrightarrow$$
$$\frac{\phi_i(N,v)}{\phi_i(N\backslash\{j\},v_{-j})}\cdot\frac{v(N)}{\sum_{k\in N}\phi_k(N,v)}\cdot\frac{\sum_{k\in N\backslash\{j\}}\phi_k(N\backslash\{j\},v_{-j})}{v_{-j}(N\backslash\{j\})}$$
$$=\frac{\phi_j(N,v)}{\phi_j(N\backslash\{i\},v_{-i})}\cdot\frac{v(N)}{\sum_{k\in N}\phi_k(N,v)}\cdot\frac{\sum_{k\in N\backslash\{i\}}\phi_k(N\backslash\{i\},v_{-i})}{v_{-i}(N\backslash\{i\})}$$
$$\overset{\text{Eq (7.4)}}{\Leftrightarrow}$$
$$\frac{\sum_{k\in N\backslash\{j\}}\phi_k(N\backslash\{j\},v_{-j})}{v_{-j}(N\backslash\{j\})}=\frac{\sum_{k\in N\backslash\{i\}}\phi_k(N\backslash\{i\},v_{-i})}{v_{-i}(N\backslash\{i\})}.$$

This completes the proof. □

Let $\phi:\mathcal{G}^N\to\mathbb{R}^N$ be an inefficient value. For any game $\langle N,v\rangle\in\mathcal{G}^N$, we define the *distribution relative difference* of $\phi$ in $\langle N,v\rangle$ as

$$\frac{\sum_{k\in N}\phi_k(N,v)}{v(N)}.$$

Proposition 7.8 can be identified as saying that the proportional-normalized value of an inefficient positive value satisfying balanced relative contributions in $\mathcal{P}^N$ satisfies balanced relative contributions in $\mathcal{P}^N$, if and only if, for any positive game $\langle N,v\rangle\in\mathcal{P}^N$ and any pair of players $i$, $j\in N$, the inefficient positive value's distribution relative differences in $\langle N,v_{-j}\rangle$ and $\langle N,v_{-i}\rangle$ are equal.

## 7.5 Conclusions

In this chapter, we started a discussion on some profile theory to combine cooperative games and revenue problems. The started point is to normalize those inefficient values in cooperative games with some revenue sharing rules of revenue problems. As in any revenue problem, the investments should not be negative, we only considered inefficient positive values for positive games.

For any positive game and inefficient positive value, we built the corresponding revenue problem based on the worth of the grand coalition and the inefficient value, and then used the proportional revenue sharing rule and the equal surplus allocation to normalize the inefficient positive value. Assuming that the inefficient positive value satisfies some usual properties in monotonic games, we discussed these properties of the normalized values in positive games. In the future, we will have a deeper discussion on this combination.